Discrete Geometric Analysis and its Applications

Date : 7 January, 2022 (Fri.) — 9 January, 2022 (Sun.)

Venue: Online (Zoom)

09 Jan. (Sun.)

17:50 - 17:55 Hisashi Naito (Chiar of scientific committee, Nagoya Univ.)
Closing

Broadening discrete surface classes

Wayne Rossman

Kobe University

Discrete surface theory is a central topic within the more general newly-developing research field called discrete differential geometry, abbreviated as DDG, and one of the important goals in this topic is to expand the classes of surfaces that can be studying while still maintaining the underlying mathematical structures.

We will mention a number of approaches to this goal, and then, in more detail, describe one particular approach, as follows: while discrete isothermic surfaces have been defined and have an underlying integrable-systems-based mathematical structure, we can extend this to include discrete surfaces that are not themselves isothermic, but do have associated sphere congruences which are isothermic in a certain sense, an extension that still allows us to keep the underlying mathematical structure.

This topic has origins in the work of Darboux, Bianchi, Guichard and Demoulin in the early 20th century, leading to progress in smooth surface geometry that we now recognize has close relations to integrable systems. Isothermic surfaces were introduced by Bour, Guichard surfaces by Guichard, and Omega surfaces by Demoulin. All of these have dual surfaces, which then guide us to their rich transformation theory as developed by Bianchi, Calapso, Darboux, Eisenhart and Guichard. Further, we find they all fit naturally into the more symmetric settings of Moebius and Lie sphere geometries, where the surfaces can be regarded as Legendre maps and Legendre immersions.

From this smooth theory of Omega surfaces and their subclasses, we can then obtain a satisfying theory for discrete Omega surfaces that replicates the smooth case in almost every detail, as we will describe in this talk.

This talk is based on joint work with Fran Burstall, Joseph Cho, Udo Hertrich-Jeromin and Mason Pember.

Deformation space of circle patterns

Wai Yeung Lam

Tsinghua University

William Thurston proposed regarding the map induced from two circle packings with the same tangency pattern as a discrete holomorphic function. A discrete analogue of the Riemann mapping is deduced from Koebe-Andreev-Thurston theorem. One question is how to extend this theory to Riemann surfaces and relate classical conformal structures to discrete conformal structures. Since circles are preserved under complex projective transformations, we consider circle packings on surfaces with complex projective structures. Kojima, Mizushima and Tan conjectured that for a given combinatorics the deformation space of circle packings is diffeomorphic to the Teichmueller space. In this talk, we explain how the cotangent Laplacian is used to prove the conjecture for the torus case and its connection to Weil – Petersson geometry.

Discrete developable surfaces and their singularities

Kosuke Naokawa

Hiroshima Institute of Technology

A surface generated by a smooth motion of a line in Eulidean 3-space $\mathbb{R}^3$ is called a ruled surface, and if its Gaussian curvature vanishes identically, then it is called developable. Developable surfaces are locally isometric to $\mathbb{R}^2$ with the standard metric such as planes, cones, cylinders and tangential surfaces. This property gives a natural idea for discretizing developable surfaces. In fact, a 'discrete' motion of a line, that is, a sequence of lines in $\mathbb{R}^3$ is called a dicrete developable surface if any adjacent two lines of the sequence lie in a plane in $\mathbb{R}^3$, as in the following figures:

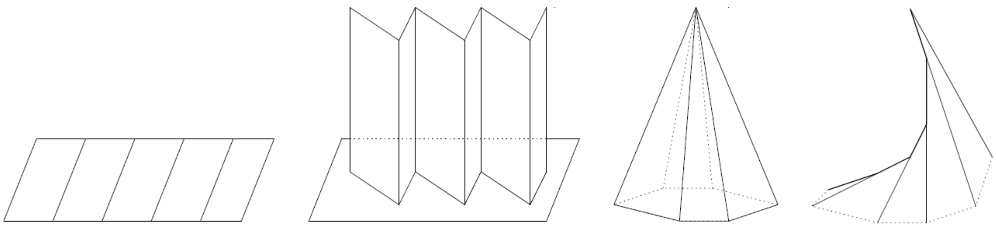

Fig. Disrete versions of a plane, cylindrical surface, cone and tangential surface, reslpectively.

In this talk, we give several results related to topologies and singularities of discrete developable surfaces. This project is based on a joint work with Chirstian Müller (TU-Wien).

Discrete timelike minimal surfaces

Masashi Yasumoto
Institute of Mathematics for Industry, Kyushu University
744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan
m-yasumoto@imi.kyushu-u.ac.jp

Based on integrable systems approach, Bobenko and Pinkall described a discrete version of isothermic surfaces. In the smooth case, isothermic surfaces include important classes of surfaces such as quadrics, surfaces of revolution, minimal surfaces, and nonzero constant mean curvature surfaces. So discrete minimal surfaces in the 3-dimensional Euclidean space were originally studied as special class of discrete isothermic surfaces, and various links to differential geometry, complex analysis, and integrable systems were found.

In this talk we focus on discrete timelike surfaces in 3-dimensional Minkowski space. They have similar properties to discrete isothermic surfaces, and in the special case there are several properties that do not appear in the case of discrete isothermic surfaces. Using them, we introduce that discrete timelike minimal surfaces admit two types of Weierstrass-type representations. As an application, we can show all the discrete timelike minimal surfaces that are not necessarily discrete timelike can be described by a discrete version of null curves.

On a discretization of confocal quadrics: Geometric parametrizations a integrable systems and incircular nets

Alexander Bobenko

TU Berlin

We propose a discretization of classical confocal coordinates. It is based on a novel characterization thereof as factorizable orthogonal coordinate systems. Our geometric discretization leads to factorizable discrete nets with a novel discrete analog of the orthogonality property. The theory is illustrated with a variety of examples in two and three dimensions. These include confocal coordinate systems parametrized in terms of Jacobi elliptic functions. Connections with incircular nets and elliptic billiards are established.

Convergence for discrete harmonic maps between nonpositively curved surfaces

Jonah Gaster

University of Wisconsin-Milwaukee

Roughly speaking, a map between Riemannian manifolds is *harmonic* if it is unreasonably well balanced, in that values at points are closer than they should be to averages of values at nearby points. Harmonic maps have proven to be enormously useful in many different branches of mathematics, in particular in the setting of hyperbolic surfaces and Teichmüller theory. I'll (briefly) discuss some of the background in the smooth setting, and focus on the problem of discretization in the setting of Riemannian surfaces.

Obtaining convergence of discrete harmonic maps to a smooth harmonic map is a subtle task, which we achieve in the setting of nonpositively curved surfaces together with some mild conditions on the discretization regime. Curiously, the proof of convergence involves two pieces of analysis which are absent in the smooth setting: 1. a uniform estimate of strong convexity for the discrete energy functional, and 2. an improvement of the usual comparison between the L^2- and L^∞-metrics on the space of discrete maps that holds near the discrete harmonic map.

This is joint work with Brice Loustau and Léonard Monsaingeon.

Percolation on crystal lattices and covering monotonicity of percolation clusters

Tatsuya Mikami*

Percolation theory is a branch of probability theory that describes the behavior of clusters. This theory has its origin in applied problems as exemplified by immersion in porous stone, represented by the following model: each edge (bond) in a lattice X is assumed to be open with the same probability $p \in [0,1]$, independently of all other edges. This model is called the *bond percolation* model and has been of great interest regarding the *critical probability* $p_c(X) \in [0,1]$, the point at which an infinite cluster of open edges appears. As one method for evaluating the critical probability, the paper [1] gives the covering monotonicity of the critical probabilities: for a free action $G \curvearrowright X$ of a group G, the comparison $p_c(X) \leq p_c(X/G)$ holds for the quotient graph X/G.

In order to observe the relationships between lattices and the shape of clusters in detail, this talk will consider the *first passage percolation* (FPP) model on a periodic realization $\Phi : X \to \mathbb{R}^d$ of a d-dimensional crystal lattice X. This model is a time evolution version of the bond percolation model: each edge e in X is assigned a random passage time $t_e \geq 0$ independently, and consideration is given to the behavior of the percolation region $B(t)$, which consists of those vertices that can be reached from the origin within a time $t > 0$. As a generalization of the result in the cubic lattice model [2], the normalized region $B(t)/t$ converges to some deterministic set $\mathcal{B}$, called the *limit shape*.

In this talk, the covering monotonicity of the limit shapes will be given as follows: Let $P : \mathbb{R}^d \to W$ be the orthogonal projection onto some suitable subspace $W \subset \mathbb{R}^d$. Then the projection $P \circ \Phi(X)$ coincides with the image $\Phi_1(X_1)$ of a periodic realization $\Phi_1 : X_1 \to W$ of some crystal lattice X_1 obtained as a quotient graph of X (see [4, Section 7.2]). Under this setting, the following comparison holds for the two limit shapes $\mathcal{B}_1$, $\mathcal{B}$ of X_1, X, respectively.

Theorem 1. $\mathcal{B}_1 \subset P(\mathcal{B})$.

This talk is based on the paper [3].

References

[1] I. Benjamini and O. Schramm, *Percolation beyond $\mathbf{Z}^d$, many questions and a few answers*, Electron. Comm. Probab. **1** (1996), no. 8, 71–82, DOI 10.1214/ECP.v1-978. MR1423907

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[3] T. Mikami, *Covering monotonicity of the limit shapes of first passage percolation on crystal lattices* (2021), available at `arxiv:2009.11679`.

[4] T. Sunada, *Topological crystallography*, Surveys and Tutorials in the Applied Mathematical Sciences, vol. 6, Springer, Tokyo, 2013. With a view towards discrete geometric analysis. MR3014418

*Department of Mathematics, Kyoto University, E-mail:mikami1642@gmail.com

On the large deviation principle for persistence diagrams of random cubical filtration

Shu KANAZAWA *

Persistent homology, which is a main tool in the growing field of topological data analysis, allows us to describe the multiscale topological features in various data. For example, given a (high-dimensional) grayscale image data, we can construct a filtration of cubical sets (i.e., the union of pixels or higher-dimensional voxels) in an appropriate way. The persistent homology can capture the birth and death times of high-dimensional holes such as loops and cavities in the filtration. By plotting each birth-death pair into two-dimensional parameter space, we get a useful descriptor of the multiscale topological features in the grayscale image data, the so-called persistence diagram. In application, it is important to examine the effect of randomness on persistence diagrams since data contain some noise.

With the above motivation in mind, we consider persistence diagrams of random cubical filtrations. A random cubical filtration is an increasing family of random cubical sets, which are the union of randomly generated higher-dimensional unit cubes with integer coordinates. The objective of this work is to investigate the asymptotic behavior of the (random) persistence diagrams of a random cubical filtration model as the window size tends to infinity. Recently, the strong law of large numbers for the persistence diagrams was proved by Hiraoka, Miyanaga, and Tsunoda, which states that the persistence diagram converges vaguely to a deterministic measure almost surely.

In this talk, we are interested in the decay rate of the probability that the persistence diagram is far from the deterministic limiting measure. We show large deviation principles for Betti numbers, persistent Betti numbers, and the histograms generated by counting the birth-death pairs falling in each fine rectangular region. The key tool for the proofs is a general large deviation principle for regular nearly additive processes, established by Seppäläinen and Yukich [1]. Time permitting, we will also discuss the ongoing work on how to provide the large deviation principle for the persistence diagrams themselves.

This talk is based on joint work with Yasuaki Hiraoka, Jun Miyanaga, and Kenkichi Tsunoda.

References

[1] T. Seppäläinen and J. Yukich, *Large deviation principles for Euclidean functionals and other nearly additive processes*, Probab. Theory Related Fields **120** (2001), 309–345.

*Kyoto University Institute for Advanced Study, Kyoto University.
E-mail address: `kanazawa.shu.2w@kyoto-u.ac.jp`

Tension tensors of harmonic nets

Hiroki Kodama

Tohoku University / iTHEMS, RIKEN

For a given harmonic net on n-dimensional Euclidean space, we define a square matrix of order n. We call it the tension tensor. We introduce the properties of the tension tensor. We describe when a harmonic net becomes standard by using the tension tensor. We also show that the covariance of a random walk on a harmonic net is consistent with the tension tensor.

The energy minimizing discrete harmonic maps and its application to closed surface

Toru Kajigaya

Tokyo Sci University

A map from a weighted finite graph into a smooth Riemannian manifold is called a discrete harmonic map if it is a stationary point of the discrete Dirichlet energy. The notion of discrete harmonic maps was introduced in the study of geodesic triangulations of surfaces and a mathematical study of crystal in nature, and is still studied in various directions. In this talk, we will consider the existence of least energy discrete harmonic maps. In particular, we will discuss the (non-)existence of local minimizer in a simply-connected positively curved manifold, and the existence of a solution of the "double" minimization problem for hyperbolic closed surfaces. A part of this talk is based on a joint work with Ryokichi Tanaka (Kyoto).

An extension of the Bloch-Floquet theory to the Heisenberg group and its applications

Atsushi Katsuda
Faculty of Mathematics, Kyushu University

The Bloch-Floquet theory are popular tools for the investigation of materials with periodic structures. For example, we can show that the spectrum of periodic Schrödinger operators have band structures. In the context of this talk, this was applied to the following problems in the case of abelian extensions:

(1) A geometric analogue of the Chebotarev density theorem for prime closed geodesics in a compact Riemannian manifold with negative curvature

(2) A long time asymptotic expansion of the heat kernels of covering manifolds of compact Riemannian manifolds.

In this talk, we shall extend the above results by applying our version of the Bloch-Floquet theory for the Heisenberg group. Our method is based on a combination of the representation theory for discrete Heisenberg groups especially due to Pytlik and that of the Heisenberg Lie group.

As a by-product, we also give another mathematical explanation of the semi-classical asymptotic expansion formula for the spectrum of the Harper operator due to Wilkinson, which is originally done by Helffer-Sjöstrand.

The spectrum of the Harper operator is described by the following figure, which is called the celebrated Hofstadter butterfly,

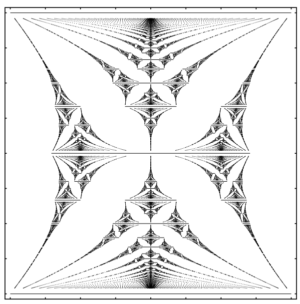

Figure 1: the Hofstadter's butterfly (created by Hisashi Naito, Special thanks to Nogizaka 46)

Discrete minimal surfaces for generating 3D fillet structures

Konrad Polthier

FU Berlin

Modelling volumetric shapes with surface meshes is a core problem in 3d-manufacturing. In this presentation we combine two centrals aspects, the topological structure of 3d-volumes and the geometric filling with surface meshes based on discrete minimal surfaces.

Classification of grain boundaries in Euclidean space

Kazutoshi Inoue[1]

Advanced Institute for Materials Research, Tohoku University

(joint work with Kazuaki Kawahara[2], Mitsuhiro Saito[2], Motoko Kotani[1],

and Yuichi Ikuhara[1,2])

Solid-state materials have been used by polycrystalline form, and their macroscopic properties strongly depend on defects by which crystalline structures are disordered. Grain boundary is especially important among them, which is an interface formed by two adjacent crystal grains.

For an n-dimensional lattice L in $\mathbb{R}^n$, $R \in SO(n)$ is said to be a coincidence rotation if the intersection $\mathrm{L} \cap R\mathrm{L}$ forms a sublattice of full rank. The sublattice $\mathrm{L} \cap R\mathrm{L}$ is called the CSL if the index $\Sigma := [\mathrm{L} : \mathrm{L} \cap R\mathrm{L}] \in \mathbb{Z}$ is finite. The CSL is the maximal sublattice contained in both L and $R\mathrm{L}$. For a coincidence isometry R and an affine hyperplane Π passing through the origin with a normal vector $\boldsymbol{n}$, Π is called a grain boundary for $\mathrm{L}_1 \cup \mathrm{L}_2$ if $\mathrm{L}_1 = \{\ \boldsymbol{p} \in \mathrm{L}; \langle \boldsymbol{p}, \boldsymbol{n} \rangle < 0\ \}$ and $\mathrm{L}_2 = \{\ \boldsymbol{p} \in R\mathrm{L}; \langle \boldsymbol{p}, \boldsymbol{n} \rangle \geqq 0\ \}$ with respect to the Euclidean inner product $\langle , \rangle$.

Although CSL grain boundaries have been classified by the index Σ, the current method may fail to list all the grain boundaries. In this talk, we propose a systematic classification method for grain boundaries by using the 2-dimensional lattice matching which is useful in designing grain boundaries. In the 2-dimensional case, the CSL theory can be formulated by the theory of quadratic field. The problem of 2-dimensional lattice matching can be deduced to the calculation of ideals of integer rings.

[1]Advanced Institute for Materials Research, Tohoku University, Katahira 2-1-1, Aoba-ku, Sendai; 980-8577

[2]Institute of Engineering Innovation, School of Engineering, The University of Tokyo, 2-11-16, Yayoi, Bunkyo-ku, Tokyo; 113-8656

Normalized discrete Ricci flow and community detection

Lin Yong

Tsinghua University

We introduce the Ricci flow equations of Ollivier-Lin-Lu-Yau curvature defined on weighted graphs. We prove the existence and uniqueness theorem for solutions to a continuous time normalized Ricci flow. We also do the graph partition using the discrete Ricci flow method.

Structural Mesh Generation Based on Computational Conformal Geometry

Na Lei

Dalian University of Tech.

Generating meshes with regular structure plays a fundamental role in computational mathematics. Regular hexahedral mesh generation is called the holy grid problem in computational mechanics. Intensive research efforts have been spent on it for tens of years. Although there are many heuristic methods in practice, the theoretic foundation still remains widely open. Recently, we have established a theoretic framework for quadrilateral mesh generation based on conformal geometry. Basically, we have discovered the intrinsic relation between quad-meshes and meromorphic differentials on Riemann surfaces. This framework is simple, elegant but powerful. It can answer many fundamental problems, that no other methods could shed a light. For examples, it can show the existence of quad-meshes with special properties, estimate the dimension of quad-meshes with constraints, specify the geometric relations among the singular vertices of quad-meshes. More importantly, it gives a simple algorithm for high quality quad-mesh generation based on Riemann-Roch and Abel-Jacobi theorems. Furthermore, the quad-meshes based on Strebel differential can leads to hexahedral mesh generation for volumes.